

## DESCRIPTIVE STATISTICS

As a starting point, we compute the descriptive statistics. The sample size is  $n$ , and the sample estimates of the average and of the standard deviation are:

Arithmetic Average (mean)

$$\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$$

Moments

$$m_k = \frac{\sum_{i=1}^n (X_i - \bar{X})^k}{n} \quad k = 0, 1, 2, \dots$$

Standard Deviation

$$s = \sqrt{m_2}$$

## NORMALITY TEST

As a first approach we use the following three tests.

1) The test of **skewness** is done for the null hypothesis  $H_0$ : normality, versus the alternative hypothesis  $H_1$ : non-normality due to skewness. We compute the skewness coefficient  $\sqrt{b_1}$  and its associated normal approximation  $Z(\sqrt{b_1})$  as follows (D'Agostino, 1970; D'Agostino and Stephens, 1986):

$$\sqrt{b_1} = \frac{m_3}{\sqrt{m_2^3}}$$

$$Y = \sqrt{b_1} \sqrt{\frac{(n+1)(n+3)}{6(n-2)}}$$

$$\beta_2(\sqrt{b_1}) = \frac{3(n^2 + 27n - 70)(n+1)(n+3)}{(n-2)(n+5)(n+7)(n+9)}$$

$$W^2 = -1 + \sqrt{2(\beta_2(\sqrt{b_1}) - 1)}$$

$$\delta = \frac{1}{\sqrt{\ln W}}$$

$$\alpha = \sqrt{\frac{2}{(W^2 - 1)}}$$

$$Z(\sqrt{b_1}) = \delta \ln \left[ \sqrt{\left(\frac{Y}{\alpha}\right)^2 + 1} + \frac{Y}{\alpha} \right]$$

2) The test of **kurtosis** is performed for the null hypothesis  $H_0$ : normality, versus the alternative hypothesis  $H_1$ : non-normality due to kurtosis. We compute the kurtosis coefficient  $b_2$  and its associated normal approximation  $Z(b_2)$  as follows (Anscombe and Glynn, 1983; D'Agostino and Stephens, 1986):

$$b_2 = \frac{m_4}{m_2^2}$$

$$E(b_2) = \frac{3(n-1)}{n+1}$$

$$\text{Var}(b_2) = \frac{24n(n-2)(n-3)}{(n+1)^2(n+3)(n+5)}$$

$$x = \frac{b_2 - E(b_2)}{\sqrt{\text{Var}(b_2)}}$$

$$\sqrt{\beta_1(b_2)} = \frac{6(n^2 - 5n + 2)}{(n+7)(n+9)} \sqrt{\frac{6(n+3)(n+5)}{n(n-2)(n-3)}}$$

$$A = 6 + \frac{8}{\sqrt{\beta_1(b_2)}} \left[ \frac{2}{\sqrt{\beta_1(b_2)}} + \sqrt{\frac{4}{\beta_1(b_2)} + 1} \right]$$

$$Z(b_2) = \frac{1 - \frac{2}{9A} - \sqrt[3]{\frac{1 - \frac{2}{9A}}{1 + x \sqrt{\frac{2}{A-4}}}}}{\sqrt{\frac{2}{9A}}}$$

3) The **omnibus** test (D'Agostino and Pearson, 1973) presents a statistic that combines the above two tests and produces an omnibus test of normality. The null hypothesis is  $H_0$ : normality, versus the alternative hypothesis  $H_1$ : non-normality due to either skewness or kurtosis. The test statistic is:

$$K^2 = Z^2(\sqrt{b_1}) + Z^2(b_2)$$

When the population is normal, it has approximately a chi-square distribution with 2 degrees of freedom. Computational details are presented by D'Agostino et al. (1990).

As a second approach, we use the concept of probability plots. It is based upon a graphical presentation of the transformed data that will be approximately lying on a straight line if the distribution is normal. Deviations from linearity correspond to various types of non-normality, such as skewness, kurtosis, outliers or censoring in the data. Using the approximation of Blom (1958) for each data  $X_i$ , where  $X_i$  is the  $i^{\text{th}}$  ordered observation from the ordered sample  $X_1 \leq X_2 \leq \dots \leq X_n$ , we get a point of coordinates

$$\left( \Phi^{-1}\left(\frac{i - \frac{3}{8}}{n + \frac{1}{4}}\right), X_i \right)$$

where  $\Phi$  is the Laplace function:

$$\Phi(u) = \int_{-\infty}^u \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

Similarly, using the approximation of Tukey (1962), we get the coordinates

$$\left( \Phi^{-1}\left(\frac{i - \frac{1}{3}}{n + \frac{1}{3}}\right), X_i \right)$$

while for the approximation of Van der Waerden (Lehmann, 1975) we get

$$\left( \Phi^{-1}\left(\frac{i}{n+1}\right), X_i \right)$$

## AUTOCORRELATION FUNCTION

When the data are evenly spaced we can compute the normalized autocorrelation function,  $R(k)$ . If the time series is completely uncorrelated, its normalized autocorrelation function is the so-called  $\delta$ -Dirac function. As a lack-of-fit test we compute the Q-statistic (Box and Pierce, 1970; Ljung and Box, 1978):

$$Q = n(n+2) \sum_{k=1}^{k=m} (n-k)^{-1} R^2(k)$$

$$m \cong \sqrt{n}$$

which follows approximately a  $\chi^2$  distribution with  $m$  degrees of freedom.

## LOMB PERIODOGRAM

When the data are evenly spaced:

$$X_i = X((i-1)\Delta), \quad i = 1, 2, \dots, n \quad (27)$$

$$f_c = \frac{1}{2\Delta}$$

where  $\Delta$  is the sampling interval and  $f_c$  is the Nyquist critical frequency, we can perform a Fourier analysis (Brigham, 1974). The data contain complete information about all spectral components up to the Nyquist frequency, and scrambled or aliased information about components at frequencies larger than the Nyquist frequency. Because in our case the data are not evenly spaced,  $X_i = X(t_i)$ , we cannot perform a Fourier analysis. Lomb (1976), using the results of Barning (1963) and Vanicek (1971), developed an alternative method to deal with the case of unevenly spaced data. Scargle (1982) refined it. The Lomb normalized periodogram, spectral power as a function of frequency  $\omega = 2\pi f$ , is defined as:

$$P_N(\omega) = \frac{1}{2s^2} \left\{ \frac{[\sum_{i=1}^n (X_i - \bar{X}) \cos \omega(t_i - \tau)]^2}{\sum_{i=1}^n \cos^2 \omega(t_i - \tau)} + \frac{[\sum_{i=1}^n (X_i - \bar{X}) \sin \omega(t_i - \tau)]^2}{\sum_{i=1}^n \sin^2 \omega(t_i - \tau)} \right\}$$

where  $\tau$  is defined by:

$$\tan(2\omega\tau) = \frac{\sum_{i=1}^n \sin(2\omega t_i)}{\sum_{i=1}^n \cos(2\omega t_i)}$$

The constant  $\tau$  is a kind of offset that makes  $P_N(\omega)$  completely independent of shifting all the  $t_i$ 's by a constant. It makes the Lomb periodogram identical to what is obtained if one estimates the harmonic content of a data set, at a given frequency  $\omega$ , by linear least-squares (Lomb, 1976) fitting to:

$$X(t) = A \cos(\omega t) + B \sin(\omega t)$$

Therefore, the Lomb periodogram weights the data on a "per point" basis instead of on a "per time-interval" basis. We assume that our data are the result of a deterministic component and of an independent white Gaussian noise process. The null hypothesis is  $H_0$ : the data are independent white Gaussian random values, versus the alternative hypothesis  $H_1$ : the data have a deterministic component, too. The "normalization" of the Lomb periodogram, through its denominator  $s^2$ , means that at any particular  $\omega$  and in the case of the null hypothesis,  $P_N(\omega)$  has an exponential probability distribution with unit average (Scargle, 1982). Therefore, if we scan  $M$  independent frequencies, the probability that none gives values larger than  $z$  is  $(1 - e^{-z})^M$ , and therefore the significance level of any peak in  $P_N(\omega)$  is:

$$P(> z) = 1 - (1 - e^{-z})^M$$

Horne and Baliunas (1986) give results from extensive Monte Carlo experiments for determining  $M$  in various cases. In general  $M$  depends on the number of frequencies sampled, the number of data points  $n$ , and their detailed spacing. It turns out that  $M$  is nearly equal to  $n$  when the data are approximately evenly spaced, and when the sampled frequencies "fill" the frequency range up to the Nyquist frequency. In our case we have searched up to twice the Nyquist frequency and therefore  $M = 2n$ .

## SENSITIVITY ANALYSIS

High-leverage points are those for which the value of the independent variable is, in some sense, far from the rest of the data (Hocking and Pendleton, 1983). The leverage,  $h_i$  is defined as:

$$h_i = [x_i^T (X^T W X)^{-1} x_i] w_i$$

where  $\mathbf{X}$  is the  $\mathbf{N} \times \mathbf{r}$  matrix containing the values of the  $\mathbf{r}$  independent variables,  $\mathbf{x}_i$  is a column vector containing the elements of the  $i$ -th row of  $\mathbf{X}$ ,  $\mathbf{W}$  is a diagonal matrix containing the weights associated with the  $\mathbf{N}$  experimental points, and  $()^{-1}$  denotes the generalized inverse of a matrix. In our case, because of the intercept,  $\mathbf{r} = 2$ . Chatterjee and Hadi (1988) point out that the leverage can be viewed as the equivalent number of observations that determine the  $i$ -th prediction. Huber (1981) suggests a 0.2 critical value. According to this rule, special attention should be given to observations whose predicted values are determined by an equivalent of 5 or fewer observations. Because  $\mathbf{r}/\mathbf{N}$  is the average value, Hoaglin and Welsch (1978) suggested  $2\mathbf{r}/\mathbf{N}$  as a critical value.

Let define the residual for the  $i$ -th experimental point,  $e_i$ , as the difference between the predicted and experimental values. An outlier is a point exhibiting a residual in absolute value by far greater than the rest. The outlier is a peculiarity and it should be submitted to a particularly careful examination to see if the reason for this peculiarity can be determined. Sometimes the outlier provides information due to the fact that it arises from an unusual combination of circumstances that may be of special interest to the researcher and requires further investigation rather than rejection (Draper and Smith, 1981). Outliers are identified via statistical measures based on residuals. Chatterjee and Hadi (1988) introduced the internally studentized residual (also called the standardized residual) defined as the usual residual divided by its estimated standard deviation:

$$ISR_i = e_i \sqrt{\frac{w_i}{s^2 (1 - h_i)}}$$

and the externally studentized residual (also called the jackknife residual) estimated when that point is deleted from the variance estimation as:

$$ESR_i = e_i \sqrt{\frac{w_i}{s_i^2 (1 - h_i)}}$$

where:

$$s_i^2 = \frac{(N - r) s^2 - \frac{w_i e_i^2}{1 - h_i}}{N - r - 1}$$

An approximate critical value for ISR is  $\sqrt{[(N-r)F/(N-r-1+F)]}$  where F is the  $100(1-\alpha/N)$ th percentile of the  $F_{1,N-r-1}$  distribution. Because ESR has a student distribution with  $(N-r-1)$  degrees of freedom, a reasonable critical value choice would be 2, while 3 seems to be a conservative one (Chatterjee and Hadi, 1988).

Influential points are those observations that excessively influence the fitted regression equation as compared to other observations in the data set. Measures for detecting influential points are commonly based on the omission approach ( i.e. they measure changes in the parameters estimates or predicted values when the  $i$ -th data point is deleted from the analysis ). Cook's distance (Cook, 1977) measures the change in the estimated regression coefficients:

$$D_i = \frac{w_i h_i e_i^2}{r s^2 (1 - h_i)^2}$$

It combines two measures, giving information about high-leverage points and outliers. Cook and Weisberg (1982) refer to it as the potential of the  $i$ -th observation in the determination of the regression parameters. As critical values, Cook (1977) suggests the percentiles of the  $F_{r,N-r}$  distribution. Welsch and Kuh (1977) introduced a similar measure, DFFITS, based on the externally studentized residual:

$$DFFITS_i = e_i \sqrt{\frac{w_i h_i}{s_i^2 (1 - h_i)^2}}$$

Belsley et al. (1980) recommend using  $2\sqrt{(r/N)}$  as a cut-off value, but  $2\sqrt{[1/(N-1)]}$  would be a more appropriate choice (Chatterjee and Hadi, 1988).

We point out that outliers need not be influential points, influential points need not be outliers, and while points with large residuals are not desirable, a small residual does not imply that the corresponding observation is a typical one. It is expected that some individual data points may be flagged as outliers, high-leverage or influential points. Any point falling into one of these categories should be carefully examined for accuracy (transcription error, gross error), relevancy or special significance (abnormal market conditions). Outliers should always be scrutinized carefully. Points

with high-leverage that are not influential do not cause any problem, but points with high leverage that are influential should be looked at carefully. If no unusual circumstances are found, these points should not be deleted as a routine matter (Chatterjee and Hadi , 1988). To get an idea of the sensitivity of the data, the parameters should be estimated with and without the above-mentioned points.

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